

Bäcklund transformation for non-relativistic Chern-Simons vortices

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Abstract.

A Bäcklund transformation yielding the static non-relativistic Chern-Simons vortices of Jackiw and Pi is presented.

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1. Introduction

In the non-relativistic ‘Chern-Simons’ version of the Abelian Higgs model of Jackiw and Pi [1-2-3], the scalar field, Ψ , is described by the *gauged, planar non-linear Schrödinger equation*

$$(1.1) \quad i\partial_t \Psi = \left[-\frac{1}{2}(\vec{\nabla} - i\vec{A})^2 + A^0 - g\Psi^*\Psi \right] \Psi.$$

(We work in our units where $\hbar = m = e = 1$). Here the ‘electromagnetic’ field, associated with the vector potential (A^0, \vec{A}) , is assumed to satisfy the Chern-Simons field-current identity

$$(1.2) \quad B = \epsilon^{ij} \partial_i A^j = -\frac{1}{\kappa} \varrho, \quad E^i = -\partial_i A^0 - \partial_t A^i = \frac{1}{\kappa} \epsilon^{ij} J^j,$$

($i, j = 1, 2$), where $\varrho = \Psi^* \Psi$ and $\vec{J} = (-i/2)[\Psi^* \vec{D} \Psi - \Psi(\vec{D} \Psi)^*]$ denote the particle density and the current, respectively. Explicit multivortex solutions have only been found so far for the special value of $g = \pm 1/\kappa$: for static fields, the second-order field equations above can be reduced to the first-order ‘self-dual’ system $(D_x \pm iD_2)\Psi = 0$. In a suitable gauge and away from the zeros of ϱ , this becomes the Liouville equation

$$(1.3) \quad \Delta \log \varrho = \pm \frac{2}{\kappa} \varrho.$$

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Regular solutions arise by choosing the upper (resp. lower) sign for $\kappa < 0$ (resp. for $\kappa > 0$),

$$(1.4) \quad \varrho = 4|\kappa| \frac{|f'|^2}{[1 + |f|^2]^2},$$

where $f \equiv f(z)$ is a meromorphic function on the complex plane. The remaining fields are expressed in terms of the particle density as

$$(1.5) \quad A^0 = \frac{1}{2|\kappa|} \varrho, \quad \vec{A} = \frac{1}{2} \vec{\nabla} \times \log \varrho + \vec{\nabla} \omega, \quad \Psi = \sqrt{\varrho} e^{i\omega},$$

where ω is chosen so that \vec{A} is regular at the zeros of ϱ .

For $g = \pm 1/\kappa$, *all* static solutions are self-dual. This can be proved using the conformal invariance of the system [2].

The problem of integrability of the full, time-dependent second-order system (1.1-2) was examined by Lévy et al. [4] and by Knecht et al. [5], who found that it was *not* in general integrable. In [5] it was shown in particular, that the full system fails to pass the Painlevé test, as extended to partial differential equations by Weiss, Tabor and Carnevale (WTC) [6].

The point is that the WTC method — when it works — has the additional bonus to provide Bäcklund transformations for generating solutions. In this paper, we take advantage of this to construct, in the static case and for $g = \pm 1/\kappa$, Bäcklund transformations allowing us to rederive all static solutions.

2. A Bäcklund transformation

Write $\Psi = \sqrt{\varrho} e^{i\omega}$ and introduce, following Knecht et al. [5], the new variables

$$(2.1) \quad \begin{aligned} \varrho &= |\kappa|^3 f^2, & A^0 - \partial^0 \omega &= -\frac{\kappa^2}{2} w \\ A_1 - \partial_1 \omega &= -\kappa u, & A_2 - \partial_2 \omega &= -\kappa v, \\ x_1 &= \frac{x}{|\kappa|}, & x_2 &= \frac{y}{|\kappa|}, \end{aligned}$$

in terms of which the static CS Eqns read ⁽¹⁾

$$(2.2) \quad \begin{aligned} u_y - v_x &= -f^2, \\ w_x &= 2vf^2, & w_y &= -2uf^2, \\ f_{xx} + f_{yy} &= -2\epsilon^{-1} f^3 - f(w - u^2 - v^2), \end{aligned}$$

where $\epsilon = 1/g|\kappa|$. The WTC method [6], [5] amounts to developing the fields into a generalized Laurent series,

$$(2.3) \quad \begin{aligned} u &= \sum_{k=0}^{\infty} u_k \Phi^{k-p_u}, & v &= \sum_{k=0}^{\infty} v_k \Phi^{k-p_v}, \\ w &= \sum_{k=0}^{\infty} w_k \Phi^{k-p_w}, & f &= \sum_{k=0}^{\infty} f_k \Phi^{k-p_f}, \end{aligned}$$

where $\Phi = 0$ is the ‘singular manifold’. Inserting these expressions into the eqns. of motion fixes the values of the p ’s and provides us with recursion relations, except for some particular values in k called ‘resonances’,

⁽¹⁾ One actually gets one more relation, which corresponds to the continuity equation and appears here as a consistency condition.

when consistency conditions have to be satisfied. In detail, for $k = 0$ we find, consistently with Knecht et al. [5], that $p_u = p_v = p_f = 1$, $p_w = 2$ and

$$(2.4) \quad \begin{aligned} u_0 &= -\epsilon \Phi_y, & v_0 &= \epsilon \Phi_x, \\ w_0 &= \epsilon^2(\Phi_x^2 + \Phi_y^2), & f_0^2 &= -\epsilon(\Phi_x^2 + \Phi_y^2). \end{aligned}$$

Resonances occur for $k = 1, 2$ and 4 . For $k = 1$ we get

$$(2.5) \quad \begin{aligned} 2f_0f_1 &= -u_{0y} + v_{0x}, \\ 2f_0^2v_1 + \Phi_xw_1 + 4v_0f_0f_1 &= w_{0x}, \\ -2f_0^2u_1 + \Phi_yw_1 - 4u_0f_0f_1 &= w_{0y}, \\ -2f_0u_0u_1 - 2f_0v_0v_1 + f_0w_1 - 6(\Phi_x^2 + \Phi_y^2)f_1 &= (\Phi_{xx} + \Phi_{yy})f_0 + 2(\Phi_xf_{0x} + \Phi_yf_{0y}), \end{aligned}$$

From the first of these equations we deduce, using Eq. (2.4)

$$(2.6) \quad f_1^2 = -\epsilon \frac{(\Delta\Phi)^2}{4(\Phi_x^2 + \Phi_y^2)}.$$

The remaining system of three equations has a vanishing determinant. Consistency requires hence

$$(2.7) \quad \epsilon^2 = 1 \quad \text{i.e.} \quad g = \frac{1}{|\kappa|}.$$

Then u_1 and v_1 can be expressed as a function of w_1 , which is arbitrary.

$k = 2$ is again a resonance value: the l.h.s. of

$$(2.8) \quad \begin{aligned} \Phi_yu_2 - \Phi_xv_2 + 2f_0f_2 &= -u_{1y} + v_{1x} - f_1^2, \\ 2f_0^2v_2 + 4v_0f_0r_2 &= w_{1x} - 2(f_1^2v_0 + 2v_1f_0f_1), \\ -2f_0^2u_2 - 4u_0f_0r_2 &= w_{1y} + 2(f_1^2u_0 + 2u_1f_0f_1), \\ -2f_0u_0u_2 - 2f_0u_0v_2 + f_0w_2 &= \\ -6(\Phi_x^2 + \Phi_y^2)f_2 &= -(f_{0xx} + f_{0yy}) - 6\epsilon^{-1}f_1^2f_0 - w_1f_1 \\ &\quad + f_0(u_1^2 + v_1^2) + 2f_1(u_0u_1 + v_0v_1). \end{aligned}$$

has vanishing determinant. Consistency requires hence the same to be true for the r.h.s., which only happens when

$$(2.9) \quad w_1 = -2\epsilon f_0f_1 = -\Delta\Phi.$$

The arising of the constraint (2.9) shows already that even the static system (2.2) fails to pass the Painlevé test of WTC, which would allow an arbitrary w_1 . We can, nevertheless, continue our search for finding solutions constrained to satisfy (2.9). Then u_2 , v_2 and w_2 are expressed using the arbitrary function f_2 . (Condition (2.9) is below related to *self-duality*).

Inserting w_1 into (2.5) we get, with the help of (2.4) and (2.6)

$$(2.10) \quad \begin{aligned} 2(\Phi_x^2 + \Phi_y^2)v_1 - \epsilon \Delta\Phi\Phi_x &= -\epsilon(\Phi_x^2 + \Phi_y^2)_x, \\ 2(\Phi_x^2 + \Phi_y^2)u_1 + \epsilon \Delta\Phi\Phi_y &= \epsilon(\Phi_x^2 + \Phi_y^2)_y, \end{aligned}$$

while the last equation is identically satisfied. Thus

$$\begin{aligned}
 (2.11) \quad u &= -\epsilon (\log \Phi)_y + u_1 + u_2 \Phi + \dots, \\
 v &= \epsilon (\log \Phi)_x + v_1 + v_2 \Phi + \dots, \\
 w &= -\Delta \log \Phi + w_2 + w_3 \Phi + \dots, \\
 f^2 &= \epsilon \Delta \log \Phi + (2f_0 f_2 + f_1^2) + (2f_1 f_2 + 2f_0 f_3) \Phi + \dots
 \end{aligned}$$

Now we try to *truncate* these infinite series by only keeping terms of order less or equal to zero,

$$(2.12) \quad u_k = v_k = w_{k+1} = f_k \equiv 0 \quad \text{for } k \geq 2.$$

Inserting these relations into the first equation of (2.8), we find that the r. h. s. vanishes. Hence

$$(2.13) \quad u_{1y} - v_{1x} = -f_1^2.$$

From the other eqns. of (2.8) we get, for the Ansatz (2.12) and using the constraint (2.9),

$$(2.14) \quad f_{1x} = -\epsilon v_1 f_1, \quad \text{and} \quad f_{1y} = \epsilon u_1 f_1$$

and

$$(2.15) \quad w_2 = -\epsilon f_1^2.$$

Note that if f_1 is not identically zero, then Eq. (2.14) implies

$$(2.16) \quad \left(\frac{f_{1x}}{f_1} \right)_x + \left(\frac{f_{1y}}{f_1} \right)_y = -\epsilon f_1^2$$

which is the *Liouville equation*. Requiring the constraint (2.9) means therefore reducing the second-order equation (2.2) to a first-order system.

So far we have only considered the terms $k = 0, 1, 2$. The consistency of our procedure follows from the verification, using the formulæ given in Ref. [5], that all remaining equations, including the compatibility condition for the resonance value $k = 4$, are identically satisfied. The case $k = 3$ we shows in particular that the fields f_1, u_1, v_1, w_2 satisfy the equations (2.2) we started with; they provide us therefore with a “seed solution” in our Bäcklund transformation.

Collecting our results and returning to the physical variables, cf. Eq. (2.1), we have proved the following. Let $\rho \sim f_1^2, \vec{a} \sim (u_1, v_1), a^0 \sim w_2$ be any “seed solution”,

$$(2.17) \quad \kappa \vec{\nabla} \times \vec{a} = -\rho, \quad \vec{\nabla} \times \rho = -2\epsilon (\text{sign} \kappa) \vec{a} \rho, \quad a^0 = \epsilon \frac{1}{2|\kappa|} \rho,$$

such that

$$\begin{aligned}
 (2.18) \quad (\Delta \Phi)^2 &= -4 \frac{\epsilon}{|\kappa|} \rho [(\partial_1 \Phi)^2 + (\partial_2 \Phi)^2], \\
 2(\text{sign} \kappa) [(\partial_1 \Phi)^2 + (\partial_2 \Phi)^2] \epsilon_{ij} a_j + \epsilon \Delta \Phi \partial_i \Phi &= \epsilon \partial_i [(\partial_1 \Phi)^2 + (\partial_2 \Phi)^2]
 \end{aligned}$$

($i = 1, 2$). Then the Bäcklund transformation

$$(2.19) \quad \begin{cases} \varrho = \epsilon |\kappa| \Delta \log \Phi + \rho, \\ \vec{A} = \epsilon (\text{sign} \kappa) \vec{\nabla} \times \log \Phi + \vec{a} + \vec{\nabla} \omega, \\ A^0 = \frac{1}{2} \Delta \log \Phi + a^0 \end{cases}$$

(where $\varrho \sim f^2, \vec{A} \sim (u, v), A^0 \sim w$) provides us with a new set of solutions for (1.1-2).

This follows either from the proof above, or can be directly verified.

If $\rho \neq 0$, the first two equations in (2.17) reduce to the Liouville equation for ρ , while \vec{a} and a^0 are expressed as in (1.5). The seed solution is hence necessarily *self-dual*.

3. The construction of solutions

The seed solution may *not* be a physical one: a judicious choice may simplify a great deal the equations to be solved. Below we obtain in fact the *general solution* (1.4) by choosing ρ to *vanish*.

$$(3.1) \quad \rho \equiv 0$$

In detail, let us assume that (3.1) holds. Then Eq. (2.17) is satisfied by $a^0 \equiv 0$ and \vec{a} any curl-free vectorpotential. Similarly, the upper equation in (2.18) now requires that Φ solve the Laplace equation

$$(3.2) \quad \Delta \Phi = 0.$$

Introducing the complex notations $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$, $\partial = \frac{1}{2}(\partial_1 - i\partial_2)$, $\bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$, the solution of (3.2) is given by $\Phi(z, \bar{z}) = f(z) + g(\bar{z})$, where $f(z)$ is analytic and $g(\bar{z})$ is anti-analytic. With this choice, ϱ is seen to be real if $g(\bar{z}) = 1/\overline{f(z)}$. Hence

$$(3.3) \quad \Phi(z, \bar{z}) = f(z) + \frac{1}{\overline{f(z)}}.$$

Positivity of ϱ requires finally to set $\epsilon = 1$. In conclusion, the particle density is

$$(3.4) \quad \varrho = |\kappa| \Delta \log \Phi = |\kappa| \Delta \log [1 + |f|^2],$$

i.e., the general solution (1.4) of the Liouville eqn.

The second Eqn. in (2.18) allows us to express the vector potential of the seed solution as

$$(3.5) \quad \vec{a} = -\frac{1}{2}(\text{sign } \kappa) \vec{\nabla} \times \log [(\partial_1 \Phi)^2 + (\partial_2 \Phi)^2] \equiv -\frac{1}{2}(\text{sign } \kappa) \vec{\nabla} \times \log \frac{|f'|^2}{f^2},$$

whose curl indeed vanishes, as required for consistency. Note that \vec{a} combines with the first term in the \vec{A} -equation of (2.19) to yield the curl of $\log \varrho$, cf. Eq. (1.5). Finally, $A^0 = \varrho/2|\kappa|$ follows from (2.17) and (2.19).

For example, for each fixed $0 \neq c \in \mathbf{C}$,

$$(3.6) \quad \Phi = (z/c)^N + (\bar{z}/\bar{c})^{-N}$$

is a rotationally invariant solution of the Laplace equation, which yields the well-known radial solution

$$(3.7) \quad \varrho = \frac{4N^2|\kappa|}{r^2} \left(\left(\frac{r}{r_0} \right)^N + \left(\frac{r_0}{r} \right)^N \right)^{-2}$$

($r_0 = |c|$ [1-3]). The first term in the \vec{A} -equation of (2.19) is

$$(3.8) \quad (\text{sign } \kappa) i \frac{2N}{\bar{z}} \frac{1}{1 + |z|^{2N}},$$

while the seed solution is

$$(3.9) \quad a \equiv a_1 + ia_2 = -(\text{sign } \kappa) i \frac{N+1}{\bar{z}}.$$

At the origin, the sum of these terms behaves as

$$(3.10) \quad (\text{sign } \kappa) i \frac{(N-1)}{\bar{z}},$$

so that the singularities are avoided if the phase ω ($\partial_t \omega = 0$) is chosen to be

$$(3.11) \quad \omega = (-\text{sign } \kappa) (N-1) \arg z.$$

The magnetic charge $Q \equiv \int B d^2 \vec{r}$ is conveniently calculated as

$$(3.12) \quad Q = \oint_S \vec{A} \cdot d\vec{\ell},$$

where $S \equiv S_\infty$ denotes the circle at infinity. At infinity (3.8) falls off, so that only the seed and ω terms contribute. We find

$$Q = -(\text{sign } \kappa) 2\pi(N+1) - (\text{sign } \kappa) 2\pi(N-1) = -(\text{sign } \kappa) 4\pi N,$$

as expected. More generally,

$$(3.13) \quad \Phi = \frac{\prod_{i=1}^N (z - z_i)}{P(z)} + \frac{\overline{P(z)}}{\prod_{i=1}^N (\bar{z} - \bar{z}_i)}$$

where the z_i 's are arbitrary complex numbers and $P(z)$ is a polynomial of z of degree at most $N-1$ ($P(z_i) \neq 0$), provides us with a $4N$ -parameter family of solutions with magnetic charge $Q = (-\text{sign } \kappa) 4\pi N$. This can be shown along the same lines as above.

4. Discussion

Our formulæ are equivalent to the expression in Refs. [1-3], [7]. To see this, observe that Φ is manifestly invariant with respect to the transformation $f \rightarrow 1/\bar{f}$. But ϱ is also invariant with respect to complex conjugation $f \rightarrow \bar{f}$ (cf. (3.4)), and thus also with respect to $f \rightarrow 1/f$. But for (3.13), $1/f$ is decomposed into partial fractions as

$$(4.1) \quad \frac{1}{f} = \sum_{i=1}^N \frac{d_i}{z - z_i}.$$

which is the standard choice [1-3], [7].

A Bäcklund transformation for the Liouville equation has been constructed before by D'Hoker and Jackiw [8]. Their approach also involves a solution of the Laplace equation. They need, however, to solve an additional system of coupled, first-order differential equations. Our formulæ are hence different from theirs.

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